Everywhere Regularity of Solutions to a Class of Strongly Coupled Degenerate Parabolic Systems

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Abstract

A class of strongly coupled degenerate parabolic system is considered. Sufficient conditions will be given to show that bounded weak solutions are Hölder continuous everywhere. The general theory will be applied to a generalized porous media type Shigesada-Kawasaki-Teramoto model in population dynamics.

1 Introduction

In the present paper we study the Hölder continuity of bounded weak solutions to nonlinear parabolic systems of m equations ($m \geq 2$) given by

$$u_t = \mathrm{div}(a(x,t,u)\nabla u) + f(x,t,u), \tag{1.1}$$

in a domain $Q = \Omega \times (0,T) \subset \mathbb{R}^{N+1}$, with Ω being an open subset of $\mathbb{R}^N$, $N \geq 1$. The *vector valued* functions u, f take values in $\mathbb{R}^m$, $m \geq 1$. ∇u denotes the spatial derivative of u. Here, $a(x,t,u) = (A_{ij}^{\alpha\beta})$ is a tensor in $\mathrm{Hom}(\mathbb{R}^{nm}, \mathbb{R}^{nm})$.

A weak solution u to (1.1) is a function $u \in W_2^{1,0}(Q, \mathbb{R}^m)$ such that

$$\iint_Q [-u\phi_t + a(x,t,u)\nabla u \nabla \phi]\, dz = \iint_Q f(x,t,u)\phi\, dz$$

for all $\phi \in C_c^1(Q, \mathbb{R}^m)$. Here, we write $dz = dxdt$.

The problem of regularity of bounded solutions to such systems is a long-standing problem, which has just been intensively studied in recent years. For systems with *regular* diffusion part $a(x,t,u)$, *partial* regularity results were established by Giaquinta and Struwe in [4]. However, the question of whether bounded weak solutions are Hölder continuous *everywhere* was only answered in very few situations under either a severe restriction on the dimension N of the domain Ω, $N \leq 2$, as in [6], or special structural conditions on $a(x,t,u)$ for arbitrary N (see [10, 13]).

To the best of our knowledge, such questions have not been addressed for systems like (1.1) having certain degeneracy in the tensor a. Important examples include cross diffusion systems modelling phenomena in porous media. In contrast to the single equation case (see [9]), one cannot expect in general that bounded weak solutions of (1.1) will be Hölder continuous everywhere. In a recent work [11], we investigated the question of partial regularity of (1.1) having the following structure conditions.

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(A.1) There exists a C^1 map $g : \mathbb{R}^m \to \mathbb{R}^m$, with $\Phi(u) = \nabla_u g(u)$, such that for some positive constants $\lambda, \Lambda > 0$ there hold

$$a(u)\nabla u \cdot \nabla u \geq \lambda |\nabla g(u)|^2, \quad |a(u)\nabla u| \leq \Lambda |\Phi(u)||\nabla g(u)|.$$

(A.2) (Degeneracy condition) $\Phi(0) = 0$. There exist positive constants C_1, C_2 such that

$$C_1(|\Phi(u)| + |\Phi(v)|)|u - v| \leq |g(u) - g(v)| \leq C_2(|\Phi(u)| + |\Phi(v)|)|u - v|.$$

(A.3) (Comparability condition) For any $\beta \in (0, 1)$, there exist constants $C_1(\beta), C_2(\beta)$ such that if $u, v \in \mathbb{R}^m$ and $\beta|u| \leq |v| \leq |u|$, then $C_1(\beta)|\Phi(u)| \leq |\Phi(v)| \leq C_2(\beta)|\Phi(u)|$.

(A.4) (Continuity condition) $\Phi(u)$ is invertible for $u \neq 0$. The map $a(u)\Phi(u)^{-1}$ is continuous on $\mathbb{R}^m \setminus \{0\}$. Moreover, there exists a monotone nondecreasing concave function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$, ω is continuous at 0, and

$$|a(v)\Phi(v)^{-1} - a(u)\Phi(u)^{-1}| \leq (|\Phi(u)| + |\Phi(v)|)\omega(|u - v|^2), \tag{1.2}$$

$$|\Phi(u) - \Phi(v)| \leq (|\Phi(u)| + |\Phi(v)|)\omega(|u - v|^2) \tag{1.3}$$

for all $u, v \in \mathbb{R}^m$.

Introducing the so called *A-heat approximation method*, we were able to extend the partial regularity results in [4] to the degenerate system (1.1). The main result of [11] is the following characterization of the regular sets of bounded weak solutions.

Theorem 1.1 *([11]) Let u be a bounded weak solution to (1.1) satisfying (A.1)-(A.4). Set*

$$\textit{Reg}(u) = \{(x, t) \in \Omega \times (0, T) : \ u \textit{ is Hölder continuous in a neighborhood of } (x, t)\}$$

and $\textit{Sing}(u) = \Omega \times (0, T) \setminus \textit{Reg}(u)$. Then $\textit{Sing}(u) \subseteq \Sigma_1 \bigcup \Sigma_2$, where

$$\Sigma_1 = \{(x, t) \in \Omega \times (0, T) : \liminf_{R \to 0} |(u)_{Q_R(x,t)}| = 0\},$$

$$\Sigma_2 = \{(x, t) \in \Omega \times (0, T) : \liminf_{R \to 0} ⨏\!\!⨏_{Q_R} |u - (u)_{Q_R(x,t)}|^2 \, dz > 0\}.$$

Here, for each $R > 0$, $Q_R(x, t) = B_R(x) \times (t - R^2, t)$ and $(u)_{Q_R(x,t)} = ⨏\!\!⨏_{Q_R(x,t)} u \, dz$. Moreover, $H^n(\Sigma_2) = 0$, where H^n is the n-dimensional Hausdorff measure.

Obviously, whether bounded weak solutions are Hölder continuous everywhere, that is $\text{Sing}(u) = \emptyset$, is an important question and still remains open. There are no previous results concerning everywhere regularity for general systems of the form (1.1). The results and methods in aforementioned works [6, 10, 13] for regular systems cannot apply here. New techniques and additional structure conditions will be needed. This will be the main goal of this paper.

We begin our paper, in Section 3, by considering systems like (1.1) of m equations ($m \geq 2$) and giving sufficient conditions (in addition to (A.1)-(A.4)) that guarantee everywhere regularity of bounded weak solutions. Roughly speaking, our method relies on the key assumption on the existence of a function $H(u)$. This function links the structures of the equations in a way that we can derive certain regularity of $H(u)$, which is regarded as a function in (x,t). Such regularity of $H(u)$ will be exploited later to study that of u. This technique was first introduced by us in [10] to handle the regular cases. Here, we make use of the scaled parabolic cylinders in order to reflect the degeneracy $\Phi(u)$. This idea was originally introduced in [1] to deal with scalar p-Laplacian equations. However, the case of degenerate systems needs much more sophisticated techniques. Another difficulty arises as the L^2 estimate for ∇u, derived by Giaquinta and Struwe in [4, page 443] for regular cases, is no longer available here to obtain the smallness of the average of the deviation $|u-(u)_{Q_R}|^2$ on Q_R. Direct estimates of these quantities must be rediscovered. In addition, we must also show that the system is *averagely* not too degenerate in certain scaled cylinder so that the component Σ_1 of the singular set is empty.

We demonstrate our general theory by considering a degenerate Shigesada-Kawasaki-Teramoto (SKT) model arising in population dynamics. Here, we incorporate the porous media type diffusion into the well studied regular (SKT) systems. We will give sufficient conditions on the parameters of this system such that a function H can be found; and the results of Section 3 are applicable. The existence of a function H for general regular (SKT) systems were studied in [10]. Our degenerate system (SKT) obviously necessitates a different H, but some calculations in [10] are reusable here. The new choice of H in this work also greatly simplifies many complicated calculations in [10].

We would like to remark that we assume no presence of ∇u in the lower order term f in (1.1) for the sake of simplicity. In fact, in [11] and this present work, we could allow f to depend on ∇u, and to have growth like $\varepsilon|\Phi(u)|^2|\nabla u|^2$ for sufficiently small $\varepsilon > 0$. The proof for this case is similar, with an exception of some minor technical modifications.

The paper is organized as follows. In Section 2, we introduce our notations, hypotheses and main theorems. We study the general system (1.1) in Section 3. Section 4 is devoted to the degenerate (SKT) system and concludes our paper.

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2 Notations and main results

Throughout this paper, Ω is a bounded domain in $\mathbb{R}^N$. For a scalar function $h(x,t)$, with $(x,t) \in \mathbb{R}^{N+1}$, its spatial (resp. temporal) derivative with respect to the x (resp. t)variable is denoted by ∇h (resp. $\partial h/\partial t$ or h_t). If $u = (u_1, \ldots, u_m)$ is a vector valued function, then $\nabla u = (\nabla u_1, \ldots, \nabla u_m)$. If H is a function in u, then $H_u = \nabla_u H = (\partial_{u_1} H, \ldots, \partial_{u_m} H)$.

For a given set $X \subset \mathbb{R}^n$ we denote by $|X|$ its n dimensional Lebesgue measure. We write $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$, the ball centered at x_0 with radius R. For a measurable bounded X, we denote the average of a given measurable function h over X by

$h_X = \frac{1}{|X|}\int_X h(x)dx$.

In our proof, $C, C_1, \ldots$ will denote various constants whose values change from line to line but are independent of the solutions in question. For $a, b \geq 0$, we also write $a \sim b$ if there are positive constants C_1, C_2 such that $C_1 a \leq b \leq C_2 a$.

In the sequel, we first consider a bounded weak solution u to (1.1) on $\Omega \times (0,T)$ and the following conditions.

(H.1) There exists a C^2 real function $H(u)$ defined on a neighborhood of the range of the solution u. Moreover, for some $\gamma \geq 2$ and $|u|$ small, we have $H(u) \sim |u|^\gamma$.

(H.2) There are positive constants λ_1, λ_2, and λ_3 such that

$$\begin{gathered} H_u^T a(u)\nabla u \nabla H \geq \lambda_1 |\Phi(u)|^2 |\nabla H|^2, \\ \nabla H_u^T a(u)\nabla u \geq \lambda_2 |\nabla g(u)|^2, \\ |H_u^T a(u)\nabla u| \leq \lambda_3 |\Phi(u)|^2 |\nabla H|. \end{gathered}$$

We are now in a position to state our first theorem on the everywhere regularity.

Theorem 2.1 *Given the conditions (A.1)-(A.4) and (H.1)-(H.2), bounded weak solutions to (1.1) are Hölder continuous on* $\Omega \times (0,T)$.

To illustrate this general result, we then study a system of 2 equations

$$\begin{array}{rcl} u_t & = & \nabla(\bar{P}_u(u,v)\nabla u + \bar{P}_v(u,v)\nabla v) + F(u,v), \\ v_t & = & \nabla(\bar{Q}_u(u,v)\nabla u + \bar{Q}_v(u,v)\nabla v) + G(u,v), \end{array} \tag{2.1}$$

with the following degenerate structure for some $\alpha > 0$:

$$\begin{array}{ll} \bar{P}_u = a_{11}u^\alpha + a_{12}v^\alpha, & \bar{P}_v = b_{11}u^\alpha, \\ \bar{Q}_v = a_{21}u^\alpha + a_{22}v^\alpha, & \bar{Q}_u = b_{22}v^\alpha. \end{array} \tag{2.2}$$

In this form, (2.1) is a generalized version of the well known Shigesada-Kawasaki-Teramoto model in population dynamics (see [12]). By allowing the presence of powers of u, v and dropping random diffusion terms, we take into account porous media type diffusion effects. The system becomes degenerate and has not been ever discussed in existing literature.

In applications, u, v represent population densities of the species under investigation, and thus only positive solutions are of interest. Our second result deals with the regularity of these positive weak solutions.

Theorem 2.2 *Assume* $\alpha \geq 1/2$ *and the following conditions on the coefficients of (2.1)*

$$a_{11} > a_{21}, \quad a_{22} > a_{12}, \quad (a_{11} - a_{21})(a_{22} - a_{12}) > b_{11}b_{22}, \tag{2.3}$$

$$\min\{2a_{12}a_{22}, 2a_{11}a_{21}\} \geq \max\{b_{11}(a_{11} - a_{21}), b_{22}(a_{22} - a_{12})\}. \tag{2.4}$$

If (u,v) *is a positive bounded weak solution to (2.1), then* (u,v) *is Hölder continuous everywhere.*

3 The general case

We give the proof of Theorem 2.1 in this section. For the sake of simplicity, we will assume throughout that $f(x,t,u) \equiv 0$. The presence of this term would cause no major difficulties.

By translation, we will assume that $(x_0, t_0) = (0,0)$. Fix an $\epsilon \in (0,2)$ and sufficiently small $R_0 > 0$. We consider the cylinder

$$Q(2R_0, R_0^{2-\epsilon}) = B_{2R_0}(0) \times [-R_0^{2-\epsilon}, 0] \subseteq \Omega_T.$$

Given $\rho > 0$, we will determine the positive constants θ and $\delta \in (0,1)$ and construct the following sequences:

$$R_n = \frac{R_0}{\theta^n},\ \mu_0 = \sup_{Q(2R_0, R_0^{2-\epsilon})} H(u(x,t)),\ \mu_{n+1} = \max\{\delta\mu_n, \theta R_n^\epsilon\},$$

$$\Phi_{\mu_n} = \sup\{|\Phi(u)| \,:\, H(u) \le \mu_n\},\ Q_n = B_{R_n}(0) \times [-\Phi_{\mu_n}^{-2} R_n^2, 0].$$

We also define the following function on Q_n:

$$w(x,t) := \log\left(\frac{\mu_n}{N(u)}\right),\ \text{with } N(u) = \frac{1}{\rho}(\mu_n - H(u)).$$

For each n, let $Q_n^0 = \{(x,t) \in Q_n \,:\, w(x,t)_+ = 0\}$. We consider the following two alternatives.

(A) For all integers n, we have

$$|Q_n^0| > \rho |Q_n|. \tag{3.1}$$

(B) For some integer n, we have

$$|Q_n^0| \le \rho |Q_n|. \tag{3.2}$$

Let us briefly explain how Theorem 2.1 follows from these two alternatives.

Given any $\varepsilon > 0$, we will show that $\rho = \rho(\varepsilon) > 0$ can be chosen such that if (3.2) holds for some (fixed) n, then there are fixed constants $\mu, \beta > 0$ such that

$$\sup_{Q_R} |u| \le \mu,\quad \fint\!\!\fint_{Q_R} |u - u_{Q_R}|^2\, dz \le \varepsilon\mu^2 \text{ and } |u_{Q_R}| \ge \beta\mu, \quad R = R_n/2. \tag{3.3}$$

The Hölder continuity of u then follows immediately from (3.3) and Theorem 1.1.

Otherwise, for such ρ, we suppose that (3.1) holds for all integers n. We will show that the followings are true for all integers n.

$$H(u(x,t)) \le \mu_n \quad \forall (x,t) \in Q_n, \tag{3.4}$$

$$Q_{n+1} \subseteq Q_n. \tag{3.5}$$

Arguing as in the proof of [7, Lemma 5.8], we can see that the sequence $\{\mu_n\}$ satisfies $\mu_n \le C(R_n/R_0)^\alpha$ for some $\alpha > 0$ and some constant C depending only on θ, R_0, μ_0. Due to (3.4), $H(u(x,t))$ is Hölder continuous. The assumption (H.1) then gives the Hölder continuity of $u(x,t)$.

Remark 3.1 If $H(u) \ge \sigma\mu_n$ for some $\sigma > 0$, then there is constant $C = C(\sigma) > 0$ such that $|\Phi(u)| \ge C\Phi_{\mu_n}$. Indeed, let $\Phi_{\mu_n} = |\Phi(u_0)|$ for some u_0 such that $H(u_0) \le \mu_n$. (H.1) implies that $|u|^\gamma \ge C_1(\sigma)\mu_n$ and $|u_0|^\gamma \le C_2\mu_n$. Hence, $|u_0| \le C_3(\sigma)|u|$ for some $C_3(\sigma)$. This and (A.3) give $\Phi_{\mu_n} \le C(\sigma)|\Phi(u)|$.

Alternative (A): First of all, by scaling and assuming that $\Phi_{\mu_0} \ge CR_0^\epsilon$, we can make $Q_0 \subseteq Q(2R_0, R_0^{2-\epsilon})$ so that (3.4) and (3.5) are verified for $n = 0$. Moreover, we can also assume that $\mu_n \le 1$ for all n.

Assume that (3.4) holds for some integer n. Let $R = R_n/4$ and η be a function with compact support in $Q_R = B_R \times [-\Phi_{\mu_n}^{-2}R^2, 0]$.

We test the equation of u_i by $H_{u_i}\eta/N$ and add the results to get

$$\int_\Omega \frac{\partial w}{\partial t}\eta\, dx + \int_\Omega \left[\frac{H_u^T a(u)\nabla u}{N}\nabla\eta + \frac{\nabla H_u^T a(u)\nabla u}{N}\eta + \frac{H_u^T a(u)\nabla u\nabla H}{N^2}\eta\right] dx = 0. \tag{3.6}$$

If $\eta \ge 0$, then (H.2) and the above imply

$$\int_\Omega \frac{\partial w}{\partial t}\eta\, dx + \int_\Omega \frac{H_u^T a(u)\nabla u}{N}\nabla\eta\, dx \le 0. \tag{3.7}$$

We first show that $\|w\|_{\infty,Q_R}$ can be estimated in terms of $\|w\|_{2,Q_{2R}}$. By (H.2), we have

$$\left|\frac{H_u^T a(u)\nabla u}{N}\right| \le \lambda_3|\Phi(u)|^2|\nabla w|, \quad \frac{H_u^T a(u)\nabla u}{N}\nabla w = \frac{H_u^T a(u)\nabla u\nabla H}{N^2} \ge \lambda_1|\Phi(u)|^2|\nabla w|^2.$$

We then see that the assumptions of [9, Lemma 3.3] are satisfied. Moreover, on the set $w^+ > 0$ we have $H > (1-\rho)\mu_n$. Remark 3.1 asserts that $|\Phi(u)| \ge C\Phi_{\mu_n}$ on the set $w^+ > 0$. Furthermore, since $H(u(x,t)) \le \mu_n$ on Q_n by (3.4), we have $|\Phi(u(x,t)| \le \Phi_{\mu_n}$ on Q_n. Hence, the comparability property (3.12) of [9, Lemma 3.4] is verified too. The iteration argument of [9, Lemma 3.5] then gives a constant C independent of R and Φ such that

$$\sup_{B_R\times[-\Phi_{\mu_n}^{-2}R^2,0]} w \le C(1 + \frac{1}{|Q_R|}\iint_{Q_R} (w_+)^2\, dz). \tag{3.8}$$

Next, we replace η by η^2 in (3.6) and use (H.2) to get

$$\int_\Omega \frac{\partial w}{\partial t}\eta^2\, dx + \int_\Omega |\Phi(u)|^2|\nabla w|^2\eta^2\, dx \le C\int_\Omega (|\Phi(u)|^2|\nabla w|\eta|\nabla\eta|\, dx.$$

Having established that $|\Phi(u)| \sim \Phi_{\mu_n}$ on the set $w^+ > 0$ and $meas(\{w^+ = 0\}) = meas(Q_n^0) \geq \rho|Q_n|$ by (3.1), we can follow the proof of [9, Lemma 3.6] to show that $\frac{1}{|Q_R|}\iint_{Q_R} (w_+)^2\, dz$ can be bounded by a constant independent of R and Φ. By (3.8), $\sup_{B_R\times[-\Phi_{\mu_n}^{-2}R^2,0]} w$ is also bounded by a constant, denoted by $\ln(C)$, independent of R and Φ. From the definition of w, we easily get

$$H(u(x,t)) \leq \delta\mu_n, \quad \forall(x,t) \in Q_R, \tag{3.9}$$

with $\delta = \frac{C-\rho}{C} < 1$ and depends only on ρ.

We now show that (3.5) is verified by a suitable choice of θ.

To proceed, we claim that there is a constant $C_0 = C(\delta)$ such that $\Phi_{\mu_n} \leq C_0\Phi_{\mu_{n+1}}$. Indeed, let u_1 be such that $\Phi_{\mu_n} = |\Phi(u_1)|$ and $H(u_1) \leq \mu_n$. Since $\mu_n \leq \mu_{n+1}/\delta$, we have $|u_1|^\gamma \leq C_1(\delta)\mu_{n+1}$. Hence, for some $C_2(\delta)$, we have $u_2 = C_2(\delta)u_1$ satisfying $H(u_2) \leq \mu_{n+1}$. This gives that $|\Phi(u_1)| \leq C_3(\delta)|\Phi(u_2)| \leq C_0(\delta)\Phi_{\mu_{n+1}}$, due to (A.3). Our claim then follows.

We then determine θ such that $Q_{n+1} \subseteq Q_R$. This is to say, $R_{n+1} \leq R = R_n/4$ and $\Phi_{\mu_{n+1}}^{-2}R_{n+1}^2 \leq \Phi_{\mu_n}^{-2}R^2$. To this end, we need $\theta \geq 4$ and $\Phi_{\mu_n} \leq \Phi_{\mu_{n+1}}\theta/4$. We then choose $\theta = \max\{4, 4C_0(\delta)\}$.

Therefore, $Q_{n+1} \subseteq Q_R \subseteq Q_n$ and (3.9) holds on Q_{n+1}. This shows that (3.4) continues to hold for $n+1$. By induction, we conclude that (3.4) and (3.5) hold for all integers n. Our proof is complete in this case.

Alternative B: We now have (3.2) for some n. Denote $R = R_n/4$ and

$$Q_{4R} = Q_n, \quad Q_R = B_R \times [-\Phi_{\mu_n}^{-2}R^2, 0].$$

It is easy to see that (3.2) yields

$$|Q_n^0| = |\{(x,t) \in Q_{4R} \,:\, H \leq (1-\rho)\mu_n\}| < \rho|Q_{4R}|. \tag{3.10}$$

We first derive L^p estimates for $|\nabla g|$. Test (1.1) with $H_u\eta$ to get

$$\int_\Omega \frac{\partial H}{\partial t}\eta\, dx + \int_\Omega (\nabla H_u)^T a(u)\nabla u\eta\, dx + \int_\Omega H_u^T a(u)\nabla\eta\, dx = 0. \tag{3.11}$$

Let $H_k^+ = (H(u(x,t)) - k)^+$. Replacing η in (3.11) by $H_k^+\eta^2$, we easily obtain

$$\begin{aligned}\iint_{\Omega_T} \frac{\partial(H_k^+\eta)^2}{\partial t}\, dz \quad &+ \quad \iint_{\Omega_T} [H_u^T a(u)\nabla u\nabla H_k^+\eta^2 + \nabla H_u^T a(u)\nabla u H_k^+\eta^2]\, dz \\ &\leq \quad \iint_{\Omega_T} [H_u^T a(u)\nabla u H_k^+\eta\nabla\eta + (H_k^+)^2\eta\eta_t]\, dz.\end{aligned}$$

By (H.2), this implies

$$\iint_{\Omega_T} [\lambda_1|\Phi(u)|^2|\nabla H|^2\eta^2 + \lambda_2|\nabla g(u)|^2 H_k^+\eta^2]\, dz \leq \iint_{\Omega_T} [|\Phi(u)|^2|H_k^+|^2|\nabla\eta|^2 + |H_k^+|^2|\eta_t|]\, dz.$$

We now take η to be a cut-off function with respect to the scaled cylinders Q_R, Q_{4R}. We have that $|\nabla\eta| \le \frac{1}{R}$ and $|\eta_t| \le \frac{\Phi^2_{\mu_n}}{R^2}$.

We then take $k = (1-2\rho)\mu_n$ and note that $H_k^+ \le 2\rho\mu_n$ on Q_{4R}. Moreover, because $H(u(x,t)) \le \mu_n$ on Q_{4R}, we have $|\Phi(u(x,t))| \le \Phi_{\mu_n}$. Using the fact that $|Q_R| \sim \Phi_{\mu_n}^{-2} R^{N+2}$, we obtain

$$\iint_{Q_R} |\nabla g(u)|^2 H_k^+ \; dz \le C(\rho\mu_n)^2 R^N.$$

Let $A_0 := \{(x,t) \in Q_R | H \ge (1-\rho)\mu_n\}$. Then $(H-k)_+ \ge \rho\mu_n$ on A_0. So,

$$\iint_{A_0} |\nabla g(u)|^2 \; dz \le C\rho\mu_n R^N. \tag{3.12}$$

Since $H(u) \sim |u|^\gamma$ $(\gamma \ge 2)$ and $H(u) \le \mu_n \le 1$ on Q_R, we can find C such that if $\mu = (C\mu_n)^{1/\gamma}$ then

$$\sup_{Q_R} |u(x)| \le \mu \text{ and } \mu_n \le C\mu^2. \tag{3.13}$$

By testing (1.1) with $u\eta$, it is standard to show that

$$\iint_{Q_R} |\nabla g(u)|^2 \; dz \le C\mu^2 R^N. \tag{3.14}$$

For any subset A of Q_R, Hölder's inequality gives

$$\iint_A |\nabla \vec{u}|^q \; dz \le \left(\iint_A |\nabla \vec{u}|^2 \; dz\right)^{\frac{q}{2}} |A|^{1-\frac{q}{2}}. \tag{3.15}$$

Taking $q = \frac{2N}{N+1} < 2$, $A = A_0$ and using (3.12), we obtain

$$\iint_{A_0} |\nabla g(u)|^q \; dz \le (2\rho\mu_n R^N)^{\frac{N}{N+1}} \Phi_{\mu_n}^{\frac{-2}{N+1}} R^{\frac{N+2}{N+1}} = (2\rho\mu_n)^{\frac{N}{N+1}} \Phi_{\mu_n}^{\frac{-2}{N+1}} R^{N+\frac{2}{N+1}}.$$

Similarly, we take $A = Q_R \setminus A_0$ in (3.15). Using (3.14) and also the fact that $|A| \le \rho |Q_R|$ by (3.10), we have

$$\iint_{Q_R\setminus A_0} |\nabla g(u)|^q \; dz \le (C\mu^2 R^N)^{\frac{N}{N+1}} (\rho\Phi_{\mu_n}^{-2} R^{N+2})^{\frac{1}{N+1}} = C\rho^{\frac{1}{N+1}} \Phi_{\mu_n}^{\frac{-2}{N+1}} \mu^{\frac{2N}{N+1}} R^{N+\frac{2}{N+1}}.$$

The above inequalities give us the following estimate for $|\nabla g|$:

$$\iint_{Q_R} |\nabla g(u)|^q \; dz \le C[(\rho\mu_n)^{\frac{N}{N+1}} + \rho^{\frac{1}{N+1}} \mu^{\frac{2N}{N+1}}] \Phi_{\mu_n}^{\frac{-2}{N+1}} R^{N+\frac{2}{N+1}}. \tag{3.16}$$

We now try to estimate the deviation $|u - u_{Q_R}|$. We recall the following inequality ([8, (2.10), p.45]), with $r = 1$, $p = 2$ and $m = 2N/(N+1)$, for functions u with $u_\Omega = 0$

$$\int_\Omega u^2\,dx \le C\int_\Omega |\nabla u|^{\frac{2N}{N+1}}\,dx\left(\int_\Omega |u|\,dx\right)^{\frac{2}{N+1}}.$$

Let $V(t)$ be a vector such that $g(V(t)) = g_{B_R}(u) = \frac{1}{|B_R|}\int_{B_R} g(u)dx$. The above yields

$$\begin{aligned}\iint_{Q_R} |g(u)-g(V(t))|^2\,dz &\le C\iint_{Q_R} |\nabla g(u)|^q\,dz\sup_t\left(\int_{B_R(t)} |g(u)-g_{B_R}(u)|\,dx\right)^{\frac{2}{N+1}}\\ &\le C[(\rho\mu_n)^{\frac{N}{N+1}}+\rho^{\frac{1}{N+1}}\mu^{\frac{2N}{N+1}}]\Phi_{\mu_n}^{\frac{-2}{N+1}}R^{N+\frac{2}{N+1}}(\Phi_{\mu_n}\mu)^{\frac{2}{N+1}}R^{\frac{2N}{N+1}}.\end{aligned}$$

Hence,

$$\iint_{Q_R} |g(u)-g(V(t))|^2\,dz \le \varepsilon(\rho,\mu_n,\mu)R^{N+2},$$

with $\varepsilon(\rho,\mu_n,\mu) = C[(\rho\mu_n)^{\frac{N}{N+1}}+\rho^{\frac{1}{N+1}}\mu^{\frac{2N}{N+1}}]\mu^{\frac{2}{N+1}}$.

As $|g(u)-g(V(t))| \ge C(|\Phi(u)|+|\Phi(V(t))|)|u-V(t)|$ and $H(u)\ge(1-\rho)\mu_n$ on A_0, we have $|\Phi(u)|\sim\Phi_{\mu_n}$ on the set A_0 (see Remark 3.1). Thus,

$$\Phi_{\mu_n}^2\iint_{A_0} |u-V(t)|^2\,dz \le C\varepsilon(\rho,\mu_n,\mu)R^{N+2}.$$

Let $u_{B_R} = \frac{1}{|B_R|}\int_{B_R} udx$. Because $\int_{B_r} |u-u_{B_R}|^2\,dx \le C\int_{B_r} |u-V(t)|^2\,dx$, we have

$$\begin{aligned}\Phi_{\mu_n}^2\iint_{Q_R} |u-u_{B_R}|^2\,dz &\le \Phi_{\mu_n}^2\iint_{A_0} |u-V(t)|^2\,dz+\Phi_{\mu_n}^2\iint_{Q_R\setminus A_0} |u-V(t)|^2\,dz\\ &\le C\varepsilon(\rho,\mu_n,\mu)R^{N+2}+\Phi_{\mu_n}^2\mu^2\rho|Q_R|.\end{aligned}$$

This gives $\Phi_{\mu_n}^2\iint_{Q_R} |u-u_{B_R}|^2\,dz \le C(\varepsilon(\rho,\mu_n,\mu)+\mu^2\rho)R^{N+2}$.

On the other hand, for $G = \iint_{Q_R} |u_{B_R}-u_{Q_R}|^2\,dz$, we have

$$\begin{aligned}G &\le |Q_R|\sup\nolimits_{t\in I_R}\left|\fint_{B_R} u(x,t)\,dx - \frac{1}{|I_R|}\int_{I_R}\fint_{B_R} u(x,s)\,dxds\right|^2\\ &\le |Q_R||B_R|^{-2}\sup\nolimits_{t,s\in I_R}\left|\int_{B_R} [u(x,t)-u(x,s)]\,dx\right|^2.\end{aligned}$$

From the equation of u, we have

$$\sup_{t,s\in I_R}\left|\int_{B_R} [u(x,t)-u(x,s)]\,dx\right| \le \iint_{Q_{2R}} |a(u)\nabla u\nabla\eta|\,dz \le C\frac{\Phi_{\mu_n}}{R}\iint_{Q_{2R}} |\nabla g(u)|\,dz.$$

Using the inequality $\int_A u \le \sqrt{\int_A u^2|A|}$, we argue the same way as in (3.16) to get

$$\iint_{Q_{2R}} |\nabla g(u)|\, dz \;\leq\; \sqrt{2\rho\mu_n R^N \Phi_{\mu_n}^{-2} R^{N+2}} + \sqrt{(C\mu^2 R^N)(\rho\Phi_{\mu_n}^{-2} R^{N+2})}$$
$$= C(\sqrt{\mu_n} + \mu)\sqrt{\rho}\Phi_{\mu_n}^{-1} R^{N+1}.$$

Thus, $G \leq C(\mu_n + \mu^2)\rho|Q_R|$. Putting these together and using (3.13), we have

$$⨌_{Q_R} |u - u_{Q_R}|^2\, dz \leq C(\varepsilon(\rho, \mu_n, \mu) + \mu^2\rho) + C(\mu_n + \mu^2)\rho \leq o(\rho)\mu^2.$$

Given any $\varepsilon > 0$, we can choose ρ such that

$$⨌_{Q_R} |u - u_{Q_R}|^2\, dz \leq \varepsilon\mu^2. \tag{3.17}$$

We finally show that u_{Q_R} is not small. Clearly, (3.17) yields

$$\iint_{Q_R} u^2\, dz \leq (\varepsilon\mu^2 + |u_{Q_R}|^2)|Q_R|. \tag{3.18}$$

Because $|H| \geq (1-\rho)\mu_n$ implies $|u|^\gamma \geq C\mu^\gamma$ ($\mu = (C\mu_n)^{1/\gamma}$), we see that $|u| \geq C\mu$ on A_0. Since $|Q_R \backslash A_0| \leq C\rho|Q_R|$, we also have $|A_0| \geq (1 - C\rho)|Q_R|$. Hence,

$$\iint_{Q_R} u^2\, dz \geq \iint_{A_0} u^2\, dz \geq C^2\mu^2(1 - C\rho)|Q_R|. \tag{3.19}$$

For ε and ρ sufficiently small, (3.18) and (3.19) show that $|u_{Q_R}| \geq \beta\mu$ for some constant $\beta > 0$. This, (3.13) and (3.17) give (3.3). Thus, our proof for the alternative B is complete.

4 The degenerate (SKT) system

We prove Theorem 2.2 in this section. Let us recall the system

$$\begin{array}{rcl} u_t &=& \nabla(\bar{P}_u(u,v)\nabla u + \bar{P}_v(u,v)\nabla v) + F(u,v), \\ v_t &=& \nabla(\bar{Q}_u(u,v)\nabla u + \bar{Q}_v(u,v)\nabla v) + G(u,v), \end{array} \tag{4.1}$$

with

$$\begin{array}{ll} \bar{P}_u = a_{11}u^\alpha + a_{12}v^\alpha, & \bar{P}_v = b_{11}u^\alpha, \\ \bar{Q}_v = a_{21}u^\alpha + a_{22}v^\alpha, & \bar{Q}_u = b_{22}v^\alpha. \end{array} \tag{4.2}$$

We also recall the following conditions stated in Theorem 2.2.

(P.1) $\alpha \geq 1/2$, $a_{11} > a_{21}$, and $a_{22} > a_{12}$. Moreover,

$$(a_{11} - a_{21})(a_{22} - a_{12}) > b_{11}b_{22}. \tag{4.3}$$

(P.2) We also assume that

$$\min\{2a_{12}a_{22}, 2a_{11}a_{21}\} \geq \max\{b_{11}(a_{11} - a_{21}), b_{22}(a_{22} - a_{12})\}. \tag{4.4}$$

Let (u,v) be a positive solution to (4.1). We denote by $\Gamma \subset \mathbb{R}^2_+$ the range of this solution. Our goal is to find a suitable function H that satisfies the condition (H.1), (H.2) such that Theorem 2.1 can apply here.

To begin, we set $\Delta(u,v) = \det \bar{a}(u,v) = \bar{P}_u\bar{Q}_v - \bar{P}_v\bar{Q}_u$ and

$$A(u,v) = \frac{1}{\sqrt{\Delta}}\bar{a}(u,v) = \begin{pmatrix} P_u & P_v \\ Q_u & Q_v \end{pmatrix}. \tag{4.5}$$

We also take $g = \sqrt[4]{\Delta} I_2(u,v)$, where I_2 is the 2×2 identity matrix. Thanks to (P.1), it is easy to see that A is a regular matrix, and that (A.1)-(A.4) are satisfied here.

We first observe that the condition (H.2) is verified if we can find a function H that satisfies the following conditions.

$$H_u^T A(u)\nabla u \nabla H \geq \lambda_1 |\nabla H|^2, \tag{4.6}$$

$$\nabla H_u^T A(u)\nabla u \geq \lambda_2 |\nabla u|^2, \tag{4.7}$$

$$|H_u^T A(u)\nabla u| \leq \lambda_3 |\nabla H|. \tag{4.8}$$

These conditions amount to the positivity of the following quadratics in $U, V \in \mathbb{R}^N$:

$$\begin{aligned} A_1 &= \left(\left(P_uH_u + Q_uH_v\right)H_u - \lambda_1 {H_u}^2\right)U^2 + \left(\left(P_vH_u + Q_vH_v\right)H_v - \lambda_1 {H_v}^2\right)V^2 \\ &\quad + \left(\left(P_vH_u + Q_vH_v\right)H_u + \left(P_uH_u + Q_uH_v\right)H_v - 2\lambda_1 H_vH_u\right)VU, \end{aligned} \tag{4.9}$$

$$\begin{aligned} A_2 &= \left(Q_uH_{uv} + P_uH_{uu} - \lambda_2\right)U^2 + \left(P_vH_{uv} - \lambda_2 + Q_vH_{vv}\right)V^2 \\ &\quad + \left(P_vH_{uu} + P_uH_{uv} + Q_vH_{uv} + Q_uH_{vv}\right)VU, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} A_3 &= \left(\lambda_3 {H_u}^2 - \left(P_uH_u + Q_uH_v\right)^2\right)U^2 + \left(\lambda_3 {H_v}^2 - \left(P_vH_u + Q_vH_v\right)^2\right)V^2 \\ &\quad + \left(2\lambda_3 H_vH_u - 2\left(P_vH_u + Q_vH_v\right)\left(P_uH_u + Q_uH_v\right)\right)VU. \end{aligned} \tag{4.11}$$

Following [10], the discriminants of A_1, A_3 will be nonpositive if the following first order equation is satisfied.

$$H_u = f(u,v)H_v, \tag{4.12}$$

where f is the solution to

$$-P_vf^2 + (P_u - Q_v)f + Q_u = 0. \tag{4.13}$$

Because $P_vQ_u > 0$, (4.13) has two solutions $\overline{f}_1, \overline{f}_2$ with $\overline{f}_1\overline{f}_2 < 0$. In what follows, we denote by $f = f(u,v)$ the positive solution of (4.13).

We first have the following simple lemma.

Lemma 4.1 *Assume that H satisfies (4.12). There exist positive numbers λ_1, λ_3 such that A_1, A_3 are positive definite.*

Proof: Following the proof of [10, Lemma 3.2], we need only choose λ_1, λ_3 such that the coefficients of U^2, V^2 in A_1, A_3 can be positive. By (4.13) and (4.12), these coefficients in A_1 can be written as

$$H_v^2 f^2 (P_v f + Q_v - \lambda_1), \quad H_v^2 (P_v f + Q_v - \lambda_1).$$

Similarly, for A_3, they are

$$H_u^2 (\lambda_3 - (P_v f + Q_v)^2), \quad H_v^2 (\lambda_3 - (P_v f + Q_v)^2).$$

Since $f > 0$, we can take $\lambda_1 = \frac{1}{2} \inf_\Gamma (P_v f + Q_v)$ and $\lambda_3 = 2 \sup_\Gamma (P_v f + Q_v)^2$. These are finite positive numbers because of (4.5) and the fact that u, v are bounded. ∎

Thus, we are left with the positivity of A_2. The rest of this section will be devoted to finding H that solves (4.12) and makes A_2 positive definite. This is also the crucial step in proving Theorem 2.2.

To proceed, we pick a solution g of the first order equation

$$g_u - f(u,v) g_v = 0, \tag{4.14}$$

and let G be any C^2 differentiable function on $\mathbb{R}$. Notice that $H(u,v) = G(g(u,v))$ is also a solution to (4.12).

Following the calculations of [10] (where we assumed no specification on P_u, P_v, Q_u, Q_v and the choice of g, G) we introduce the following quantities.

$$\alpha_2 = 4(Q_u P_v - P_u Q_v)(f_u - f_v f),$$
$$\alpha_3 = [(f_u + f f_v) P_v + f_v (Q_v - P_u)]^2 + 4(P_u Q_v - Q_u P_v) f_v^2,$$
$$\beta_2 = 4(f^2 + 1)(P_v f + Q_v), \quad \beta_3 = 4(Q_u f_v + P_u (f_u + f f_v) + P_v f_v),$$

and

$$\delta_{12} = \left(Q_u f + P_u f^2\right) g_{vv} + Q_u f_v g_v + P_u \left(f_u g_v + f f_v g_v\right),$$
$$\delta_{21} = (P_v f + Q_v) g_{vv} + P_v f_v g_v,$$
$$\delta_{11} = \delta_{22} = {g_v}^2 (Q_u + P_u f) = {g_v}^2 (P_v f^2 + Q_v f).$$

We recall the following elementary results that were derived in [10, Lemma 3.5]. We state this without a proof.

Lemma 4.2 *Let $\widetilde{H} = G(g)$ and $H = K\widetilde{H}$, where K is a constant. The discriminant of A_2 is given by*

$$\overline{\Theta}_2 = -4\lambda_2^2 + K \widetilde{\Theta}_{11} \lambda_2 + K^2 \widetilde{\Theta}_2, \tag{4.15}$$

where

$$\widetilde{\Theta}_{11} = (\beta_2 \widetilde{H}_{vv} + \beta_3 \widetilde{H}_v), \quad \widetilde{\Theta}_2 = (G')^2 \left[\frac{G''}{G'} g_v^3 \alpha_2 + (g_{vv} g_v \alpha_2 + g_v^2 \alpha_3) \right]. \tag{4.16}$$

Furthermore, the coefficients of U^2, V^2 in A_2 are given by $\delta_1 - \lambda_2, \delta_2 - \lambda_2$ with

$$\delta_1 = KG' (\frac{G''}{G'} f \delta_{11} + \delta_{12}), \quad \delta_2 = KG' (\frac{G'' \delta_{22}}{G' f} + \delta_{21}). \tag{4.17}$$

Concerning these quantities, we compute and get

$$f_u = -\frac{\partial_u P_v f^2 - f\partial_u P_u + f\partial_u Q_v - \partial_u Q_u}{\sigma(u,v)}, \quad f_v = -\frac{\partial_v P_v f^2 - f\partial_v P_u + f\partial_v Q_v - \partial_v Q_u}{\sigma(u,v)}.$$

Here $\sigma(u,v) := (2P_v f - P_u + Q_v) = \sqrt{(P_u - Q_v)^2 + 4P_v Q_u}$.

We first have the following simple facts about f.

Lemma 4.3 i) *Setting* $f_1 = \frac{b_{22}}{a_{22}-a_{12}}$, *and* $f_2 = \frac{a_{11}-a_{21}}{b_{11}}$, *we have* $f_1 \le f \le f_2$.

ii) $f_u > 0$ *and* $f_v < 0$.

iii) $f_u u + f_v v = 0$.

Proof: i) By substituting f_1, f_2 into the quadratic $-P_v X^2 + (P_u - Q_v)X + Q_u$ and simplifying, we easily find that this quadratic is positive (resp. negative) at f_1 (resp. f_2) thanks to (P.1). Since f_1, f_2 and f are positive and $-P_v < 0$, the claim follows.

ii) We note that f also satisfies (4.13) with P, Q being replaced by $\bar{P}, \bar{Q}$ so that

$$f_u = -\frac{\partial_u \bar{P}_v f^2 - f\partial_u \bar{P}_u + f\partial_u \bar{Q}_v}{\bar{\sigma}(u,v)} = f\alpha u^{\alpha-1}\frac{a_{11} - a_{21} - b_{11} f}{\bar{\sigma}(u,v)},$$

$$f_v = -\frac{-f\partial_v \bar{P}_u + f\partial_v \bar{Q}_v - \partial_v \bar{Q}_u}{\bar{\sigma}(u,v)} = -\alpha v^{\alpha-1}\frac{f(a_{22} - a_{12}) - b_{22}}{\bar{\sigma}(u,v)}.$$

From i), ii) follows.

iii) Direct calculation shows $f_u u + f_v v = (-\bar{P}_v f^2 + (\bar{P}_u - \bar{Q}_v)f + \bar{Q}_u)/\bar{\sigma}(u,v)$, which is zero by (4.13). The proof is complete. ∎

Our next step is to determine g from the (4.14), which can be solved by characteristic methods. From [2, pp. 97-99], we know that $\vec{x}(t) = (u(t), v(t))$, $z(t) = g(\vec{x}(t))$ and $\vec{p}(t) = (p_u(t), p_v(t)) = \nabla g(\vec{x}(t))$ solve the following system:

$$\begin{array}{rcl} \vec{x}'(t) &=& (1, -f), \\ \vec{p}'(t) &=& (f_u p_v, f_v p_v), \\ z'(t) &=& p_u - f p_v = 0. \end{array} \tag{4.18}$$

We choose the initial data for $x, \vec{p}$ on the line $\Upsilon = \{(u,v) : u = v > 0\}$, which is noncharacteristic, to be

$$\vec{x}(0) = (u,u); \quad p_u(0) = f(u,u), \quad p_v(0) = 1. \tag{4.19}$$

A smooth solution g of (4.14) can be found by setting g to be constant along each flow line $\vec{x}(t)$. In fact, we will define g on the line Υ by

$$g(u,v) = \int_0^u f(s,s)ds + v.$$

The following lemma provides useful properties of the solution g of the above system.

Lemma 4.4 *The followings hold for (4.18) and (4.19).*

i) *g is defined on the first quadrant $\{(u,v) : u,v>0\}$.*

ii) *There exist $C_1, C_2 > 0$ such that $C_1 \le g_v \le C_2$.*

iii) *There are positive constants C_1, C_2 such that $C_1(fu+v) \le g(u,v) \le C_2(fu+v)$.*

iv) $g_{vv} = -g_v \frac{f_v}{f}$.

Proof: i) Because f is bounded by Lemma 4.3, $\vec{x}(t)$ exists for all $t \in \mathbb{R}$. It is trivial to show that the flow lines cross every point in the first quadrant so that g is well defined on this set.

ii) Consider a characteristic curve emanating from a point (u_0, v_0) on Υ. From the first equation of (4.18) and the fact that f is bounded, we easily see that there is a constant C such that $-u_0 \le t \le Cv_0$ for $u(t), v(t)$ to be positive. From the equation for $\vec{p}$ in (4.18), we have $p_v(t) = \exp(\int_0^t f_v(u(s), v(s))ds$. We will estimate the last integral.

Consider the case $t \ge 0$. We have $u(t) \ge u_0$. The proof of ii) of Lemma 4.3 reveals that

$$|f_v(u(t), v(t))| \le \frac{C_1 v^{\alpha-1}}{\bar{\sigma}(u,v)} \le \frac{C_1 v^{\alpha-1}}{\sqrt{4b_{11}b_{22}u^\alpha v^\alpha}} \le \frac{C_2 v^{\alpha/2-1}}{u_0^{\alpha/2}}.$$

Thus,

$$\int_0^t |f_v| ds = \int_0^t |f_v| \frac{dv(s)}{-f} \le C_3 \int_0^{Cv_0} \frac{v^{\alpha/2-1}}{u_0^{\alpha/2}} dv \le C_4.$$

For $t < 0$, we use the fact that $f_v v = -f_u u$ to get

$$\int_0^t |f_v| ds \le \int_0^{|t|} \left| \frac{f_u u}{v} \right| ds \le C_5 \int_0^{u_0} \frac{u^{\alpha/2}}{v_0^{\alpha/2+1}} du \le C_6.$$

In both cases, we find $g_v = \exp(\int_0^t f_v(u(s), v(s))ds$ is bounded from above and below by positive constants, and conclude the proof of ii).

iii) Using the fact that f is homogeneous, we have

$$\begin{aligned} g(u,v) &= \int_0^1 g'(tu, tv)dt = \int_0^1 g_u(tu,tv)u + g_v(tu,tv)v dt \\ &= \int_0^1 (f(tu,tv)u + v) g_v(tu,tv)dt = (fu+v)\int_0^1 g_v(tu,tv)dt. \end{aligned}$$

This and ii) give the assertion.

iv) From the fact that $g_v = \exp(\int_0^t f_v(u(s), v(s))ds$, we obtain

$$g_{vv} = \frac{\partial}{\partial t} g_v \frac{\partial t}{\partial v} = \exp(\int_0^t f_v(u(s), v(s))ds) f_v(u(t), v(t)) \frac{1}{-f} = -g_v \frac{f_v}{f}.$$

This concludes our proof of the lemma. ∎

Using the above lemmas, we can tremendously simplify the quantities in Lemma 4.2. In fact, we have

Lemma 4.5 *Let g be the solution to (4.18) and (4.19). Then,*

$$\alpha_2 = -4(f_u - f_v f), \quad \alpha_3 = (f_u P_v + \frac{f_v Q_u}{f})^2 + 4f_v^2, \tag{4.20}$$

$$\delta_{12} = P_u f_u g_v, \quad \delta_{21} = -\frac{Q_v g_v f_v}{f}, \tag{4.21}$$

$$\widetilde{\Theta}_{11} = (4\delta_1 + 4\delta_2)/K. \tag{4.22}$$

Proof: The first identity of (4.20) holds trivially due to the fact that $P_u Q_v - P_v Q_u = 1$. We then use (4.13) and get

$$\alpha_3 = [f_u P_v + f_v(P_v f - (P_u - Q_v))]^2 + 4f_v^2 = (f_u P_v + \frac{f_v Q_u}{f})^2 + 4f_v^2.$$

Using iv) of Lemma 4.4, we obtain

$$\delta_{12} = -g_v \frac{f_v}{f}(Q_u f + P_u f^2) + Q_u f_v g_v + P_u(f_u g_v + f f_v g_v) = P_u f_u g_v,$$

$$\delta_{21} = -g_v \frac{f_v}{f}(P_v f + Q_v) + P_v f_v g_v = -\frac{Q_v g_v f_v}{f}.$$

Next, we note that

$$\begin{aligned}
\beta_2 g_{vv} + \beta_3 g_v &= -4g_v \left[(f^2+1)(P_v f + Q_v)\frac{f_v}{f} - (Q_u f_v + P_u(f_u + f f_v) - p_v f_v)\right] \\
&= -4g_v \left[f f_v((P_v f + Q_v)) + P_v f_v + Q_v \frac{f_v}{f} - f_v(Q_u + P_u f) - P_u f_u - P_v f_v\right] \\
&= 4P_u f_u g_v - 4\frac{Q_v f_v g_v}{f} = 4\delta_{12} + 4\delta_{21}.
\end{aligned}$$

We also observe that $4f\delta_{11} + 4\frac{\delta_{22}}{f} = \beta_2 g_v^2$. Therefore,

$$\widetilde{\Theta}_{11} = G'[\frac{G''}{G'}\beta_2 g_v^2 + \beta_2 g_{vv} + \beta_3 g_v] = (4\delta_1 + 4\delta_2)/K.$$

This gives (4.22) and finishes up the proof. ∎

We now investigate the quantity $\widetilde{\Theta}_2$. We first prove the following lemma.

Lemma 4.6 $g_{vv} g_v \alpha_2 + g_v^2 \alpha_3 < 0$.

Proof: By iv) of Lemma 4.4, we see that this quantity can be written as

$$\begin{aligned}
g_{vv} g_v \alpha_2 + g_v^2 \alpha_3 &= g_v^2 \left(4\frac{f_v(f_u - f f_v)}{f} + (f_u P_v + \frac{f_v Q_u}{f})^2 + 4f_v^2\right) \\
&= \frac{g_v^2}{f^2}\left(4f f_v f_u + (f f_u P_v + f_v Q_u)^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{g_v^2}{f^2}\left(4ff_vf_u + \frac{(fb_{11}f_uu^\alpha + b_{22}f_vv^\alpha)^2}{\Delta}\right) \\
&= \frac{g_v^2}{\Delta f^2}\left(4\Delta ff_vf_u - f_uuf_vv(fb_{11}u^{\alpha-1} - b_{22}v^{\alpha-1})^2\right) \\
&= \frac{g_v^2}{\Delta f^2}f_uf_v\left(4\Delta f - uv(fb_{11}u^{\alpha-1} - b_{22}v^{\alpha-1})^2\right).
\end{aligned}
$$

Since $f_uf_v < 0$, we need only to show that $4\Delta f - uv(fb_{11}u^{\alpha-1} - b_{22}v^{\alpha-1})^2 > 0$. This means that the following quadratic is negative in the range of f.

$$Af^2 - 2Bf + C \le 0, \tag{4.23}$$

where $A = b_{11}^2k^{2\alpha-1}$, $B = 2(a_{11}k^\alpha + a_{12})(a_{21}k^\alpha + a_{22}) - b_{11}b_{22}k^\alpha$, $C = b_{22}^2k$, and $k = u/v$.

Set $B' := 2a_{11}a_{21}k^{2\alpha} + 2a_{22}a_{12}$. By (4.3) it is clear that $B \ge B'$. Thus, we need only to show that

$$Af^2 - 2B'f + C \le 0. \tag{4.24}$$

Because

$$B'^2 - AC = (2a_{11}a_{21}k^{2\alpha} + 2a_{22}a_{12})^2 - b_{11}^2b_{22}^2k^{2\alpha} \ge k^{2\alpha}(4a_{11}a_{21}a_{22}a_{12} - b_{11}^2b_{22}^2) \ge 0$$

due to (4.4) and (4.3), we see that the quadratic in (4.24) has two solutions $\widetilde{f}_1, \widetilde{f}_2$.

By considering the cases $k > 1$ or $0 < k \le 1$ and using (4.4), we easily see that

$$\widetilde{f}_2 > \frac{B'}{A} = \frac{2a_{11}a_{21}k^{2\alpha} + 2a_{22}a_{12}}{b_{11}^2k^{2\alpha-1}} > \frac{a_{11} - a_{21}}{b_{11}} > f,$$

and

$$\widetilde{f}_1 < \frac{C}{B'} = \frac{b_{22}^2k}{2a_{11}a_{21}k^{2\alpha} + 2a_{22}a_{12}} < \frac{b_{22}}{a_{22} - a_{12}} < f.$$

Thus, (4.24) holds for all f in its range. The proof is complete. ∎

We are ready to give the main lemma of this section.

Lemma 4.7 *Let $G(g) = g^2$. For K sufficiently large and λ_2 sufficiently small we have $\Theta_1 < 0$.*

Proof: By Lemma 4.6, we have

$$\overline{\Theta}_2 = -4\lambda_2^2 + K(\beta_2\widetilde{H}_{vv} + \beta_3\widetilde{H}_v)\lambda_2 + K^2\widetilde{\Theta}_2 \le -4\lambda_2^2 + 4(\delta_1 + \delta_2)\lambda_2 + K^2G'G''g_v^3\alpha_2.$$

We consider the term $4(\delta_1 + \delta_2)\lambda_2 + K^2G'G''g_v^3\alpha_2$. By the choice of G, this term is

$$4K\left[\lambda_2(f\delta_{11} + 2g\delta_{12} + \frac{\delta_{22}}{f} + 2g\delta_{21}) + Kgg_v^3\alpha_2\right] = 4K\left[\lambda_2(f\delta_{11} + \frac{\delta_{22}}{f}) + I\right], \tag{4.25}$$

where $I := g\{2\delta_{12}\lambda_2 + 2\delta_{21}\lambda_2 + Kg_v^3\alpha_2\}$. We observe that

$$
\begin{aligned}
I &= g\left[2P_uf_ug_v\lambda_2 - 2\frac{Q_vf_vg_v}{f}\lambda_2 - Kg_v^3(f_u - ff_v)\right] \\
&= gf_ug_v\left[2\lambda_2P_u - Kg_v^2\right] - gf_vg_v\left[2\frac{\lambda_2Q_v}{f} - Kfg_v^2\right].
\end{aligned}
$$

Since g_v, f are bounded from below by positive constants and P_u, Q_v are bounded, we easily see that the expressions in the last two brackets are bounded by negative constants if K is sufficiently large. Furthermore, because

$$g \sim (fu + v), \quad f_u \sim \frac{u^{\alpha-1}}{\bar{\sigma}}, \quad f_v \sim \frac{v^{\alpha-1}}{\bar{\sigma}},$$

we have $I \sim -\frac{u^\alpha + v^\alpha}{\bar{\sigma}} \le -C$ for some positive constant C. On the other hand, it is clear that δ_{11}, δ_{22} are bounded. Consequently, the quantity in (4.25) is negative if λ_2 is small. Our claim then follows. ∎

Lemma 4.8 δ_1, δ_2 *are bounded from below by positive constants.*

Proof: Since $f_u > 0$ and $f_v < 0$, we see from (4.21) that δ_{12}, δ_{21} are positive. By (4.17) and the choice of G, the lemma follows. ∎

We conclude our paper by giving

The proof of Theorem 2.2: We now choose $H(u, v) = Kg^2(u, v)$, with g being the solution to (4.18), (4.19). iii) of Lemma 4.4 asserts that the condition (H.1) is satisfied with $\gamma = 2$. Lemma 4.1 shows that A_1, A_3 are positive definite. From Lemma 4.7 and Lemma 4.8, we conclude that A_2 is also positive definite if we choose K large and λ_2 small. Thus, (H.2) is verified and Theorem 2.1 applies here to conclude our proof. ∎

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